University of Toronto
Math Academy 2021

Admission Test

March 2021

1 Introduction

Answer the following questions and explain your procedure the best you can. If you could not produce a complete solution but have progress, please explain it as well. It is very important that you try to solve these questions without the aid of calculators or computers, at least for a time, so that you get a grasp of where the difficulty lies in these problems.

These questions will be evaluated on more than just correct answers. Some of them might be hard enough that you will not solve them entirely, but to explain what did you try to solve them, how far did you progress, what did you read and what did you miss is very important as well, and will be taken a lot into account.

Finally, also mind the presentation. The reader must be able to understand what you are doing. Mathematics should be written as any other text. In our case, in correct English with its regular rules. To convey ideas is very hard many times, if we let the disorder of our presentation to get into the way we make it harder for people to understand us. If you do not have ideas on how to write something look online for texts of mathematics to see them, even if you do not understand the mathematics in it. Relay on graphs, tables, drawings if you need to.

Be creative! Do your best.

2 Problems to be submitted

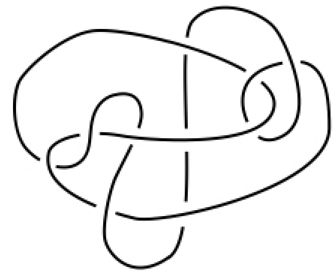
1. How many numbers from 1 to 2021, inclusive, are divisible by exactly two different prime numbers. For example, 20 is since it is divisible by exactly two: 2 and 5.

   You can verify your answer with a computer but your reasoning must not rely on computing, you must be able to reach your answer without any calculator or computer.

2. A **magic square** is an array of three times three filled in with integers, not necessarily different, such that all columns, all rows and the two diagonals all sum the same value. How many magic squares there are where all the entries are single valued digits?

   Can you answer the same question but for 4×4 magic squares?

3. The following figure is an example of a **knot**.



The idea is to start at some point and follow with the string the drawing making the string pass below or above, at each crossing, according to what the drawing indicates. At the end you finish where you started and you can tie the knot to make a knotted loop.

(a) Do the knot with a piece of a string and show a picture of it.

(b) Prove that this know is *trivial*. That is, without breaking the loop at any point you can manipulate the string so as to create a circle with the string.

Based on your explorations give a series of instructions of how to manipulate the string as to reach the trivial knot. Make an effort to do a good explanation, use drawings and tables or so to explain yourself. Make sure the reader understands you.

(c) Find a knot that is not trivial and explain why it is indeed not trivial to the best of your abilities. In general this is not an easy question to settle.

You are allowed to look for references for your examples but make sure to state your references and explain your arguments in your own words.

4. Let $C_m = \{1, 2, ..., 2^{m+1}\}$. For example $C_2 = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

(a) Let $P(x)$ be **any** quadratic polynomial. Prove that

$$P(1) + P(4) + P(6) + P(7) = P(2) + P(3) + P(5) + P(8).$$

(b) Let $P(x)$ be any polynomial of degree at most m. Prove that C_m can be broken up into two disjoint nonempty subsets A and B such that

$$\sum_{a \in A} P(a) = \sum_{b \in B} P(b)$$

(c) Give examples of these sets for the case $m = 3$.

5. A very laborious student has n activities to carry out. At some point he has to order them according to how much he likes these activities and realizes that between some activities he is unable to say what he likes or dislikes more while he is able to do it for other ones.

He writes the activities in a small sheet of paper, one activity per page and then puts activities he cannot compare in the same bag. In this way he creates several plastic bags, each with activities in them. He is able to compare the activities in one bag with those of another bag, but never among the ones in the same bag.

Once those bags are created, he puts them in order of preference (which he can, since activities in one bag can be compared with activities in another bag).

(a) For $n = 10$ activities in how many ways can this be done (i.e., what are the different possible ways of getting the final ordering of the bags). Notice that the order inside the bags is immaterial, only what is in each bag and the order of the bags matters.

(b) How would you address this problem for general n? You might want to read online about what is a generating function and an exponential generating function and explain how this tool helps to your computations.

(c) In your words, what is what makes this computational problem challenging?